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Tell-Tail Signs of Contagion

Incorporating Multidimensional Tail-Dependencies in the Valuation of Credit Derivatives

Tell-Tail Signs of Contagion: Incorporating Multidimensional Tail-Dependencies in the Valuation of Credit Derivatives

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The need for a robust and accurate representation of extreme-value risk has become increasingly acute in the wake of the credit crisis. Recent events have served all to well to highlight the disastrous consequences of inadequate in-house risk management systems and the models upon which they rely.

At least in part the genesis of the current downturn can be attributed to a fundamental mis-pricing of credit derivative instruments in a way that ignores the complexities of extreme-value ("tail") dependencies and in particular the process by which large shocks propagate through markets over time – namely contagion.

While a truthful representation of tails is of importance in assessment of risk across all asset classes it is of especial importance to the pricing of credit instruments since defaults can themselves be thought of as extreme events. As such the real value of contracts referencing multiple creditors will critically depend on their co-relationship in the tails. Unfortunately popular asset-based credit derivative (CD) models that formulate dependence relations on Student-t and other such copula are ignorant of the notion of multiple tail-risks, and in the Gaussian framework take no account of it whatsoever.

One might improve such models by admitting the possibility of structural shifts in market parameters so that returns may be perturbed from a peaceful state into a state of stress or crisis. The question as to the incidence and characteristics of these states has in part been addressed by authors such as Wang (2001) in which returns conditional on the state are assumed to be log-normal. In this normal mixture framework the parameters for each univariate margin can be related to the first four moments of the composite distribution via a simple polynomial.

By accounting for higher moments within empirical data we may better quantify the likelihood of asymmetric (skew) and extreme (kurtotic) movements. We note however that Wang fails to provide information as to the relationship between such events since within its formulation is an implicit assumption of independence of the co-occurrence of states between multiple factors. This is a crucial omission in the context of our investigation because it precludes identification of stress patterns that may arise from, and perhaps even prelude contagion effects.

Unfortunately the task of identifying these patterns is bedevilled by the curse of dimensionality: even for the simplest structural models whereby each risk factor is a mixture of two normals only, considered on a pair-wise basis each bivariate distribution is generally a mixture of four. Extending to a general n-dimensional context gives rise to a multivariate hyper-cuboid mixture of 2^n normals (see Figure1). In this manner our model allows for an unprecedented degree of flexibility in that we may explicitly specify multiple tail-dependencies within all possible sub-spaces of the multivariate. But these benefits come at the cost of a parameter space whose dimension rises exponentially with n.

This consequently poses something of a problem since to understand and quantify the movement of large shocks within and between markets one must necessarily consider systems of high dimension. Given that a model of only 20 factors requires estimation of over a million probabilities and 200 needs over a novemdecillion (10^{60}), the scope for a market-wide quantitative analysis of structural shifts would appear to be severely limited.

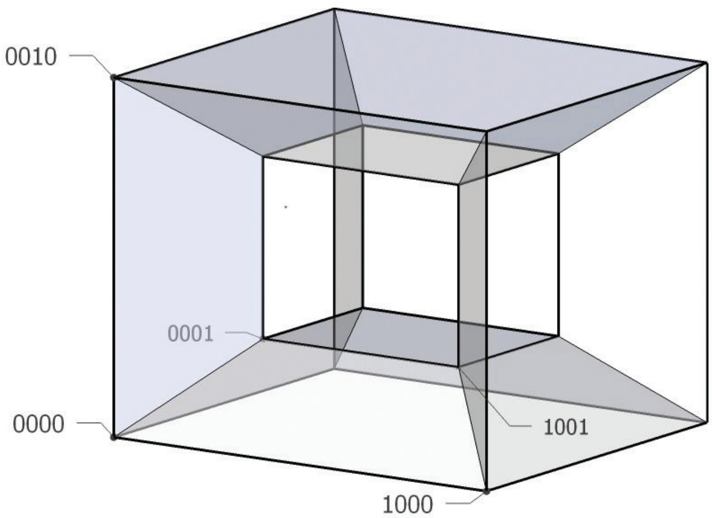

Figure 1:
Hyper-cuboid Illustration - 3D projection of a 4D multivariate

Pattern Analysis

Clearly parameterisation in its existing form is beyond the capabilities of current computing technology which would otherwise prevent any advancement in understanding of contagion and its processes. As a means to address this issue we have devised an efficient routine that recasts the high-dimensional problem as one of pattern analysis in a high-resolution image of pair-wise tail-dependencies. In doing so we may identify patterns of stress in a manner independent of dimension.

The key to this approach is to first note the geometric characteristics of the projection of a multivariate tail-dependency onto the n-by-n matrix of pair-wise tail probabilities. This matrix, denoted τ, can be thought of as containing the unconditional probabilities of joint extreme events for each unique pairing of risk factors[1]. If in a multi-dimensional context the indices of a sub-space spanned by a tail are contiguous, then the bivariate projection onto τ will be a homogeneous square block structure having reflectional symmetry about the principal diagonal and location determined by the largest and smallest of the indices. A single gap in the sequence of indices creates a cross-like gap within the original block partitioning it into four components. The complexity of the sequence of indices thus directly relates to that of the kernel structure in the τ matrix.

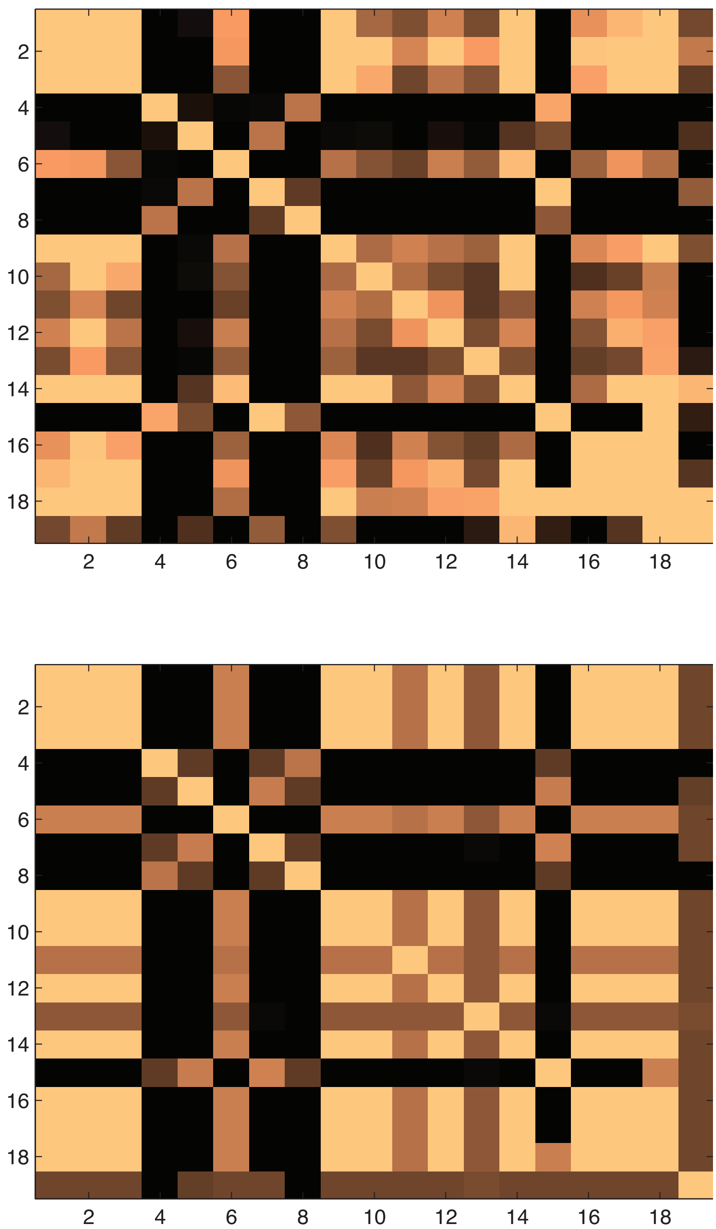

Figure 2:
Pattern Analysis Example: Target (top) and Reconstructed (bottom). Data: target implied via correlations from daily log-returns for 19 liquid iTraxx Series 5 equities between 29/06/03–29/12/03.

[1] Methods for determining τ from market data could make use of maximum likelihood estimators or as is the case here be solved so as to match observed correlations in a manner consistent with each pair of univariate specifications
(McWilliam and Loh, 2008).

In this way we may think of the parameterisation problem as one of identifying a minimal set of blocks within a matrix of observed pair-wise tail probabilities. Since the contribution of the multivariate tail probability to the bivariate tail is additive, one may overlap blocks to create complex patterns in as can be seen in Figure 2. Here we illustrate two τ matrices, the first being our target matrix as deduced from empirical data (taken from 19 of the most liquid equities referenced in iTraxx series 5), the second is the reconstructed matrix as implied from the optimised multivariate. Pattern recognition clearly captures many of the salient features of the target matrix whilst allowing massive dimension reduction of the parameterisation problem – in this instance from 524,288 to 38. The overall optimisation took 1.03 seconds using MATLAB 7.0 on an Intel Core Duo CPU @ 2.4Ghz generating an average absolute error of 0.0054.

Univariate and bivariate Kolmogorov-Smirnov tests of the 19 equities in the aforementioned iTraxx data over an 8 year period indicate that the tail-dependence model is a significantly better descriptor of returns compared with the industry standard Geometric Brownian Motion (see Figure 3 for an example).

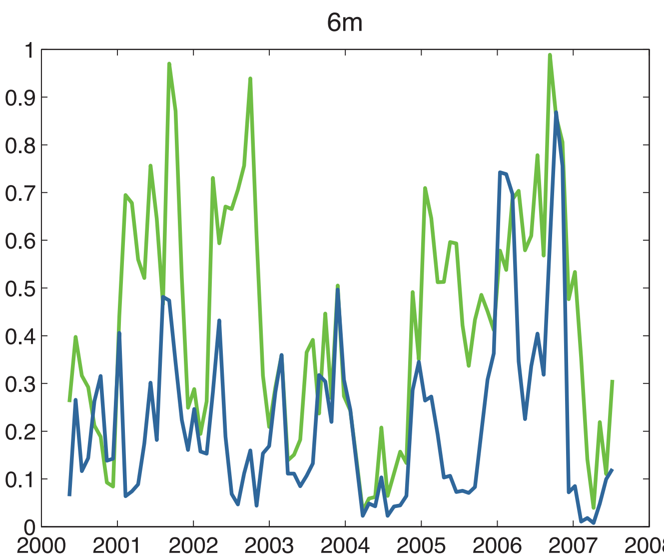

Figure 3:
Univariate KS probabilties on a 6m moving window over an 8 year period, green indicates normal mixture fit, blue a normal fit. Data: FKI Ltd Daily Log-Returns

Credit Derivative Pricing

On the basis that asset (and by proxy equity) returns contain predictive information as to the likelihood of joint default then we must surmise that failure to incorporate tail dependencies within return structures will lead to an incorrect valuation of credit derivatives.

In the context of the popular structural asset-based model for default (see Hull and White, 2001 for an example) we can make use of our hyper-cuboid mixture to extend the factor-copula approach of Li (2001) in a manner consistent with the identified tail-dependencies. A significant advantage in this regard is the ease by which we can formulate tail-consistent prices within a semi-analytical framework, potentially reducing computational effort by orders of magnitude over that of standard Monte Carlo techniques. This is of crucial importance to effective and timely risk management since the calculation of portfolio risk statistics often necessitates re-valuation of thousands of instruments over tens of thousands of market and credit scenarios across numerous time-horizons.

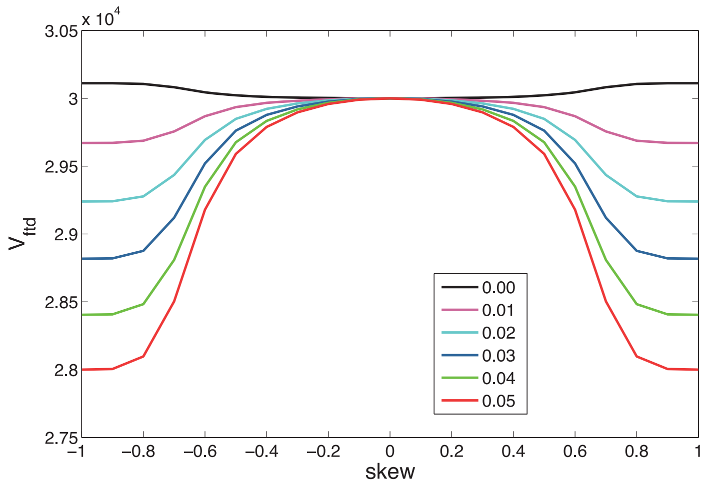

Figure 4:
FtD value as a function of skewness and tail-probability $\tau \in [0,0.05]$.

Figure 4 illustrates the strength of the effect of tail-dependency and skewness of the univariate margins on a First-to-Default Swap (FtD) value. For simplicity we consider a simple basket of two obligors only. We assume that each of the four component normals of the bivariate has zero correlation (although the overall distribution may be correlated) and a single fee of unit value is payable at T if no default occurs. The recovery rate for both obligors is set to zero as is the risk-free rate. Under these assumptions the FtD value reduces to

$$\frac{1 - S(T)}{10000 \times S(T)}$$

where $S(T)$ is the joint survival probability at time T.

One may visualise $S(T)$ as that associated with the upper right quadrant of the bivariate mixture bisected vertically and horizontally by the no-default barrier asset values (*a la Merton's* structural model). We note that the bivariate is itself reflected about the axis anti-diagonal in accordance with the sign of the skew value s which implies that the joint survival probability for s is equivalent to that of the joint default for $-s$.

Negative skew implies large negative movements in each univariate so that, all other things being equal, increasing τ reduces the likelihood of individual large negative movements, consequently increasing the probability of a joint default and joint survival. This latter point is significant for a FtD giving rise to an overall reduction of its value. As skew increases in magnitude this effect becomes more pronounced, hence the value decreases with respect to it.

Since we have chosen hazard rates such that the probability of survival of each *individual* obligor at $T=1$ is 0.5 the thresholds will be located at the median value of the each univariate which gives rise to joint survival and default probabilities that are approximately equal; thus explaining the price symmetry about the zero skew axis. In general this will not be the case.

Further Work

In this article we have discussed the importance of tail identification within a multi-dimensional setting with a view to the wider problem of contagion modelling and demonstrated its impact on the pricing of a First-to-Default Swap.

Work is currently underway within the Financial Engineering team at Misys Risk to extend the static framework presented herein making use of auto-correlation data to inform a Hidden Markov-Chain tail switching model. By creating a dynamic model for the evolution of the tails we permit a general pricing mechanism for credit derivatives in a contagious world.

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